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Uncoupling of Linear Multi-Degree-of-Freedom Damped Gyroscopic Potential Systems

This paper deals with the uncoupling of linear damped multi-degree-of-freedom gyroscopic potential systems in which the damping is taken to have a specifically chosen form. Necessary and sufficient conditions are obtained that guarantee the uncoupling of such damped systems into independent subsystems with at most two degrees-of-freedom. Along with several other results, it is shown that when the potential (stiffness) matrix of the damped system has distinct eigenvalues—a situation commonly found in civil, mechanical, and aerospace engineering, as well as in nature—the damping matrix must have this specifically chosen form for any such multi-degree-of-freedom system to be capable of being uncoupled. [DOI: 10.1115/1.4064127]

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1 Introduction

In a recent paper, the necessary and sufficient conditions have been provided for the existence of a real linear coordinate transformation that uncouples a multi-degree-of-freedom (MDOF) linear gyroscopic conservative system into a series of uncoupled subsystems, each of which has at most two degrees-of-freedom [1]. The aim of this paper is to extend these results to a broad class of damped gyroscopic potential systems and show that they too can be uncoupled into a series of at most two-degree-of-freedom subsystems. Such a decoupling into low-dimensional independent dynamical subsystems provides (a) improved physical insights into the complex behavior of such MDOF systems and (b) computational approaches that are more accurate and highly efficient, since the responses of the two-degree-of-freedom subsystems and the single-degree-of-freedom subsystems are amenable to much better physical and analytical understanding.

We consider the damped gyroscopic potential system described by

$$\tilde{M}\ddot{q} + \tilde{D}\dot{q} + \tilde{G}\dot{q} + \tilde{K}q = \tilde{f}(t) \tag{1}$$

in which $q(t)$ and $\tilde{f}(t)$ are real n-vectors, the constant n by n matrices $\tilde{M} = \tilde{M}^T > 0$, $\tilde{D} = \tilde{D}^T$, $\tilde{G} = -\tilde{G}^T$, and $\tilde{K} = \tilde{K}^T$. The dots denote differentiation with respect to time.

Using the real coordinate transformation $q(t) = \tilde{M}^{-1/2}x(t)$ and multiplying Eq. (1) by $\tilde{M}^{-1/2}$, we get

$$\ddot{x} + D\dot{x} + G\dot{x} + Kx = f(t) \tag{2}$$

where

$$D = \tilde{M}^{-1/2}\tilde{D}\tilde{M}^{-1/2} \tag{3}$$

$$G = \tilde{M}^{-1/2}\tilde{G}\tilde{M}^{-1/2} \tag{4}$$

$$K = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2} \tag{5}$$

and the n-vector

$$f(t) = \tilde{M}^{-1/2}\tilde{f}(t) \tag{6}$$

The systems described in Eqs. (1) and (2) are equivalent, and we will deal mainly with Eq. (2) in what follows. We shall refer to the matrices K, G, and D as the stiffness (potential) matrix, the gyroscopic matrix, and the damping matrix, respectively.

Consider the symmetric matrix D in Eq. (2) described by the relation

$$D = u_0 I + \sum_{i=1}^{n-1}(a_i K^i + b_i (GKG)^i) + \sum_{i=1}^{h-1} c_i G^{2i} \tag{7}$$

where u_0, a_i, b_i, and c_i are real numbers. The integer h in the upper limit of the last summation is the degree of the minimal polynomial of G^2 and equals the number of distinct eigenvalues of the matrix G^2.

We note that when $a_i = 0$ for $i = 2, \ldots, n-1$, and all the coefficients b_i and c_i in Eq. (7) are equal to zero, the corresponding damping model is the so-called "proportional damping model," and it was introduced by Rayleigh in his "The Theory of Sound" published in 1894 [2]. Also, when all the b_i and c_i are equal to zero, Eq. (7) reduces to the series

$$D = u_0 I + \sum_{i=1}^{n-1} a_i K^i \tag{8}$$

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Caughey and O'Kelly [3] showed that when all eigenvalues of the potential matrix K are distinct, the representation of the damping matrix D in Eq. (8) is both necessary and sufficient for the complete decoupling (diagonalization) of the corresponding damped (non-gyroscopic) system, i.e., of the system described by the equation $\ddot{x} + D\dot{x} + Kx = f(t)$.

We begin with the following Lemma.

LEMMA 1. *Let $K = K^T$ and $G = -G^T \neq 0$ be n by n real matrices, and let $rank(G) = 2m \leq n$. If and only if the following two conditions are satisfied*

$$KG^2 = G^2K \tag{9}$$

and

$$(KG)^2 = (GK)^2 \tag{10}$$

then there exists a real orthogonal matrix Q such that

$$Q^TGQ = \Gamma = \operatorname{diag}\left(\beta_1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \ldots, \beta_m\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, 0, \ldots, 0\right) \tag{11}$$

$$Q^TK^jQ = \Lambda^j = \operatorname{diag}(\lambda_1^j, \ldots, \lambda_n^j) \tag{12}$$

$$Q^TG^{2j}Q = \Gamma^{2j} = (-1)^j diag(\beta_1^{2j}I_2, \ldots, \beta_m^{2j}I_2, 0, \ldots, 0) \tag{13}$$

and

$$Q^T(GKG)^jQ = (\Gamma\Lambda\Gamma)^j = (-1)^j diag(\beta_1^{2j}\lambda_2^j, \beta_1^{2j}\lambda_1^j, \ldots, \beta_m^{2j}\lambda_{2m}^j, \beta_m^{2j}\lambda_{2m-1}^j, 0, \ldots, 0), \tag{14}$$

where integer $j \geq 1$, all the λ_k are real numbers, and all the β_k are nonzero real numbers.

Proof. Conditions (9) and (10) are necessary and sufficient for the existence of a real orthogonal matrix Q such that

$$Q^TKQ = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

and

$$Q^TGQ = \Gamma = \operatorname{diag}\left(\beta_1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \ldots, \beta_m\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, 0, \ldots, 0\right)$$

where the n real numbers λ_k are the eigenvalues of the matrix K, and the m imaginary pairs $\pm i\beta_k$ are the nonzero eigenvalues of G (see Ref. [1]). Then

$$Q^TK^jQ = (Q^TKQ)^j = \Lambda^j = \operatorname{diag}(\lambda_1^j, \ldots, \lambda_n^j)$$

and

$$Q^TG^2Q = (Q^TGQ)^2 = \Gamma^2 = -\operatorname{diag}(\beta_1^2I_2, \ldots, \beta_m^2I_2, 0, \ldots, 0)$$

Consequently

$$Q^TG^{2j}Q = (Q^TG^2Q)^j = \Gamma^{2j} = (-1)^j\operatorname{diag}(\beta_1^{2j}I_2, \ldots, \beta_m^{2j}I_2, 0, \ldots, 0)$$

We also have

$$Q^TGKGQ = Q^TGQQ^TKQQ^TGQ = \Gamma\Lambda\Gamma = -\operatorname{diag}(\beta_1^2\lambda_2, \beta_1^2\lambda_1, \ldots, \beta_m^2\lambda_{2m}, \beta_m^2\lambda_{2m-1}, 0, \ldots, 0)$$

and

$$Q^T(GKG)^jQ = (Q^TGKGQ)^j = (\Gamma\Lambda\Gamma)^j = (-1)^j\operatorname{diag}(\beta_1^{2j}\lambda_2^j, \beta_1^{2j}\lambda_1^j, \ldots, \beta_m^{2j}\lambda_{2m}^j, \beta_m^{2j}\lambda_{2m-1}^j, 0, \ldots, 0)$$

■

2 Main Results

We first show that when the damping matrix has the form given in Eq. (7), then under the same conditions that allow the multi-degree-of-freedom gyroscopic potential system to be uncoupled, the damped gyroscopic potential system can also be similarly uncoupled using a real coordinate transformation. Each of these uncoupled subsystems is shown to have at most two degrees-of-freedom.

Result 1. *Let $rank(G) = 2m \leq n$. If and only if*

$$KG^2 = G^2K \tag{15}$$

and

$$(KG)^2 = (GK)^2 \tag{16}$$

then there exists a coordinate change $x(t) = Qp(t)$ with $Q^TQ = I$ that decomposes Eq. (2) with the damping matrix of the form given in Eq. (7) into m decoupled two-degree-of-freedom subsystems, and $n - 2m$ decoupled single-degree-of-freedom subsystems given by

$$\ddot{p} + D_{\text{diag}}\dot{p} + \Gamma\dot{p} + \Lambda p = Q^Tf(t) \tag{17}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \tag{18}$$

$$\Gamma = \operatorname{diag}\left(\beta_1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \ldots, \beta_m\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, 0 \ldots, 0\right) \tag{19}$$

and the diagonal matrix

$$D_{\text{diag}} = \operatorname{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) = u_0I + \sum_{i=1}^{n-1}(a_i\Lambda^i + b_i(\Gamma\Lambda\Gamma)^i) + \sum_{i=1}^{h-1}c_i\Gamma^{2i}. \tag{20}$$

Proof. Using the orthogonal transformation $x = Qp$ in Eq. (2) and multiplying from the left by Q^T gives

$$\ddot{p} + Q^TDQ\dot{p} + Q^TGQ\dot{p} + Q^TKQp = Q^Tf(t)$$

It follows from Lemma 1 and Eq. (7) that an orthogonal matrix Q exists, such that

$$Q^TKQ = \Lambda, \quad Q^TGQ = \Gamma, \quad Q^TDQ = D_{\text{diag}}$$

where Λ, Γ, and D_{diag} are as in Eqs. (18), (19), and (20), if and only if conditions (15) and (16) are satisfied. We will refer to the coordinate p from here on as the principal coordinate. ■

COROLLARY 1. *Suppose that $rank(G) = 2m \leq n$ and that all nonzero eigenvalues of the matrix G are distinct. If and only if $KG^2 = G^2K$, then there exists a coordinate change $x(t) = Qp(t)$ with $Q^TQ = I$ that decomposes Eq. (2) with the damping matrix of the form (7) into m decoupled two-degree-of-freedom subsystems, and $n - 2m$ decoupled single-degree-of-freedom subsystems given by Eqs. (17)–(20).*

Proof. If all nonzero eigenvalues of the matrix G are distinct, then $KG^2 = G^2K$ implies $(KG)^2 = (GK)^2$ (see Ref. [1]), and the result follows from Result 1. ■

Remark 1. Corollary 1 points out that when the nonzero eigenvalues of G are distinct, the two necessary and sufficient conditions required for uncoupling the damped system into at most two-degree-of-freedom subsystems are replaced by a single necessary and sufficient condition that ensures such an uncoupling. ■

Example 1. Consider the system described by Eq. (2) with

$$K=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix},\quad G=\begin{bmatrix}0 & 1 & -2\\ -1 & 0 & -1\\ 2 & 1 & 0\end{bmatrix},\quad D=\begin{bmatrix}8 & -4 & 1\\ -4 & 11 & 4\\ 1 & 4 & 8\end{bmatrix} \tag{21}$$

Since G is a 3 by 3 matrix, one of its eigenvalues must be zero, and the other two eigenvalues must be purely imaginary complex conjugate pairs. Hence, its two nonzero eigenvalues are distinct.

A simple computation shows that

$$G^2=\begin{bmatrix}-5 & -2 & -1\\ -2 & -2 & 2\\ -1 & 2 & -5\end{bmatrix}$$

and it is easy to see that

$$D=9I+2K+G^2$$

Also since,

$$KG^2=-9\begin{bmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}=G^2K$$

Corollary 1 is applicable. Thus, there exists a coordinate change $x(t)=Qp(t)$ such that the system described by Eq. (2) in which the matrices K, G, and D are given in Eq. (21), decomposes into two subsystems—one with a two-degree-of-freedom and the other a single-degree-of-freedom subsystem. Indeed, the coordinate change $x=Qp$ with

$$Q=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\\ 0 & 1/\sqrt{3} & -2/\sqrt{6}\\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6}\end{bmatrix}$$

reduces this system to the form

$$\begin{bmatrix}\ddot{p}_1\\ \ddot{p}_2\end{bmatrix}+\begin{bmatrix}9\dot{p}_1\\ 3\dot{p}_2\end{bmatrix}+\sqrt{6}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix}+\begin{bmatrix}3p_1\\ 0\end{bmatrix}=\begin{bmatrix}(f_1(t)+f_3(t))/\sqrt{2}\\ (f_1(t)+f_2(t)-f_3(t))/\sqrt{3}\end{bmatrix}$$

$$\ddot{p}_3+15\dot{p}_3+3p_3=(f_1(t)-2f_2(t)-f_3(t))/\sqrt{6}$$

It is important to note that the potential matrix K of this example has two equal eigenvalues ($\lambda_1=\lambda_3=3$), while all the eigenvalues of the damping matrix D are distinct; therefore, the matrix D *cannot* be represented in the form shown in Eq. (8). ■

Corollary 2. *Let* $rank(G)=2m\le n$. *If and only if* $KG=GK$, *then there exists a coordinate change* $x(t)=Qp(t)$ *with* $Q^TQ=I$ *that decomposes Eq. (2) with the damping matrix of the form (7) into form (17) with*

$$\Lambda=\mathrm{diag}(\lambda_1I_2,\ \ldots,\lambda_mI_2,\ \lambda_{2m+1},\ \ldots,\lambda_n) \tag{22}$$

and the Γ *and* D_{diag} *same as in Eqs. (19) and (20), respectively.*

Proof. If $KG=GK$, then $KG^2=KGG=GKG=G^2K$ and $(KG)^2=(GK)^2$, and according to Result 1, the system can be decomposed into form (17)–(20) using an orthogonal transformation. In addition, $KG=GK$ requires $\lambda_1=\lambda_2,\ldots,\lambda_{2m-1}=\lambda_{2m}$ [1], and denoting repeated numbers λ_j with $\lambda_1,\lambda_2,\ldots,\lambda_m$, we get Eq. (22). Conversely, suppose an orthogonal matrix Q exists such that $Q^TKQ=\Lambda$ and $Q^TGQ=\Gamma$, with Λ and Γ given in Eqs. (22) and (19). Then, $KG=Q\Lambda\Gamma Q^T=Q\Gamma\Lambda Q^T=GK$ since it is easy to see that Λ and Γ commute now. ■

Remark 2. The commutation condition given in Corollary 2, though a single necessary and sufficient condition, is much more restrictive than the two independent conditions given in Result 1. By this, we mean that the number of matrix pairs $\{K,G\}$ that commute is generally far smaller than those that satisfy the two conditions (Eqs. (15) and (16)) given in Result 1 [1]. ■

When there exists a real orthogonal matrix Q such that $Q^TKQ=\Lambda$, $Q^TGQ=\Gamma$, and $Q^TDQ=\mathrm{diag}(\gamma_1,\ldots,\gamma_n)$, where Λ and Γ are as in Eqs. (18) and (19), respectively, we will say that the coordinate transformation $x=Qp$ transforms Eq. (2) into a *quasi-diagonal form*.

Recall that we chose a specific form for the symmetric damping matrix D in Eq. (7). We know that when the matrix D has this form, the matrices K, G, and D can be simultaneously quasi-diagonalized by a real orthogonal matrix Q (Eqs. (18)–(20)).

We next discuss the conditions under which *any* given symmetric damping matrix D that allows quasi-diagonalization of the system can be expressed in the form given in Eq. (7). We want to determine the conditions for the matrix $Q^TDQ=\mathrm{diag}(\gamma_1,\gamma_2\ldots\gamma_n)$ in which $\gamma_i's$ are arbitrary real numbers so that Q^TDQ can be expressed in the form given in Eq. (20). In other words, given an arbitrary vector $\gamma=[\gamma_1,\ldots,\gamma_n]^T$ of real numbers, we ask: under what conditions can one find all the $a_i's$, $b_i's$, $c_i's$, and u_0 so that the equation

$$\mathrm{diag}(\gamma_1,\ \ldots,\gamma_n)=u_0I+\sum_{i=1}^{n-1}(a_i\Lambda^i+b_i(\Gamma\Lambda\Gamma)^i)+\sum_{i=1}^{h-1}c_i\Gamma^{2i} \tag{23}$$

is satisfied?

Equation (23) can be rewritten as

$$\gamma=[\,\hat{1}\quad A\quad B\quad C\,]z \tag{24}$$

where $\hat{1}=[1,1,\ldots,1]^T$, $z=[u_0,\underbrace{a_1,\ldots,a_{n-1}}_{=\tilde{a}},\underbrace{b_1,\ldots,b_{n-1}}_{=\tilde{b}},\underbrace{c_1,\ldots,c_{h-1}}_{=\tilde{c}}]^T$, and

$$A=\begin{bmatrix}\lambda_1 & \ldots & \lambda_1^{n-1}\\ \lambda_2 & \ldots & \lambda_2^{n-1}\\ \ldots & \ldots & \ldots\\ \lambda_n & \ldots & \lambda_n^{n-1}\end{bmatrix}$$

$$B=\begin{bmatrix}-\beta_1^2\lambda_2 & -\beta_1^2\lambda_1 & \ldots & -\beta_m^2\lambda_{2m} & -\beta_m^2\lambda_{2m-1} & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ (-\beta_1^2\lambda_2)^{n-1} & (-\beta_1^2\lambda_1)^{n-1} & \ldots & (-\beta_m^2\lambda_{2m})^{n-1} & (-\beta_m^2\lambda_{2m-1})^{n-1} & 0 & \ldots & 0\end{bmatrix}^T$$

and

$$C=\begin{bmatrix}-\beta_1^2 & -\beta_1^2 & \ldots & -\beta_m^2 & -\beta_m^2 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ (-\beta_1^2)^{h-1} & (-\beta_1^2)^{h-1} & \ldots & (-\beta_m^2)^{h-1} & (-\beta_m^2)^{h-1} & 0 & \ldots & 0\end{bmatrix}^T$$

The linear Eq. (24) is solvable, i.e., the matrix $Q^T DQ = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ is expressible in the form in Eq. (20), if and only if

$$\mathrm{rank}\begin{bmatrix} \hat{1} & A & B & C & \gamma \end{bmatrix} = \mathrm{rank}\begin{bmatrix} \hat{1} & A & B & C \end{bmatrix}.$$

Observe that the n by n matrices $\begin{bmatrix} \hat{1} & A \end{bmatrix}$ and $\begin{bmatrix} \hat{1} & B \end{bmatrix}$ are Vandermonde matrices of eigenvalues of $K(\Lambda)$ and $GKG(\Gamma\Lambda\Gamma)$, respectively; therefore, if either K or GKG have distinct eigenvalues, then

$$\mathrm{rank}\begin{bmatrix} \hat{1} & A & B & C \end{bmatrix} = n$$

so Eq. (24) has solutions in these cases. Indeed, in the first case (see [4], for example), denoting the n by n matrix $\hat{A} := [\hat{1}\ A]$, we have

$$\det(\hat{A}) := \det([\hat{1}\ \ A]) = \prod_{1 \le j < i \le n} (\lambda_i - \lambda_j) \neq 0$$

since $\lambda_i \neq \lambda_j$ for $i \neq j$, so that $\hat{A}$ is nonsingular. Equation (24) can then be rewritten as

$$\hat{A}\begin{bmatrix} u_0 \\ \tilde{a}^T \end{bmatrix} = \gamma - B\tilde{b}^T - C\tilde{c}^T \tag{25}$$

and for any arbitrarily chosen column vectors $\tilde{b}^T$ and $\tilde{c}^T$, Eq. (25) can be solved for the vector $[u_0\ a]^T$. Thus, when the eigenvalues of K are distinct, one can always find a set of u_0, $a_i's$, $b_i's$, and $c_i's$ such that Eq. (24) (and Eq. (23)) is always satisfied. A similar argument can be made when the eigenvalues of the matrix GKG are distinct by considering the nonsingular Vandermonde matrix $\hat{B} = [\hat{1}\ \ B]$ and solving the linear equation

$$\hat{B}\begin{bmatrix} u_0 \\ \tilde{b}^T \end{bmatrix} = \gamma - A\tilde{a}^T - C\tilde{c}^T \tag{26}$$

for arbitrary column vectors $\tilde{a}^T$ and $\tilde{c}^T$, thereby obtaining a set of constants u_0, $a_i's$, , $b_i's$, and $c_i's$ that satisfy Eq. (23).

We have then the following result.

Result 2. *When either the matrix K or GKG has all distinct eigenvalues, then the conditions given by Eqs. (7), (15), and (16) are necessary and sufficient for quasi-diagonalization of Eq. (2) using an orthogonal coordinate transformation.* ∎

Example 2. To illustrate Result 2, we consider a more substantive eight-degree-of-freedom damped gyroscopic potential system described by Eq. (2) with a potential matrix given by

$$K = \begin{bmatrix} 2.2821 & -0.9874 & -0.2882 & 0.0715 & -0.0401 & -0.1114 & -0.1044 & -0.0517 \\ -0.9874 & 1.6587 & 0.5137 & -0.4269 & 0.0047 & 0.2059 & -0.3842 & -0.7003 \\ -0.2882 & 0.5137 & 2.7527 & -0.5283 & 0.1151 & -0.3265 & 0.3665 & -0.4207 \\ 0.0715 & -0.4269 & -0.5283 & 2.0280 & 0.3005 & -0.1207 & -0.0345 & -0.6098 \\ -0.0401 & 0.0047 & 0.1151 & 0.3005 & 2.0820 & -0.1580 & -0.4949 & -0.5872 \\ -0.1114 & 0.2059 & -0.3265 & -0.1207 & -0.1580 & 1.4961 & 0.1293 & -0.4578 \\ -0.1044 & -0.3842 & 0.3665 & -0.0345 & -0.4949 & 0.1293 & 2.7341 & -0.0716 \\ -0.0517 & -0.7003 & -0.4207 & -0.6098 & -0.5872 & -0.4578 & -0.0716 & 2.9662 \end{bmatrix} \tag{27}$$

whose eigenvalues are distinct, a gyroscopic matrix given by

$$G = \begin{bmatrix} 0 & 0.1678 & -0.9405 & -0.2660 & -0.5625 & -0.3012 & 0.3524 & 0.8074 \\ -01678 & 0 & 1.2850 & -0.4362 & 0.2102 & 0.0295 & -0.4523 & 0.3484 \\ 0.9405 & -1.2850 & 0 & -0.1005 & -0.1472 & -0.5767 & 0.4015 & 0.3302 \\ 0.2660 & 0.4362 & 0.1005 & 0 & -0.0959 & -0.2556 & 0.8034 & -0.2116 \\ 0.5625 & -0.2102 & 0.1472 & 0.0959 & 0 & -0.0094 & 0.0537 & 0.6806 \\ 0.3012 & -0.0295 & 0.5767 & 0.2556 & 0.0094 & 0 & 1.1843 & -0.9966 \\ -0.3524 & 0.4523 & -0.4015 & -0.8034 & -0.0537 & -1.1843 & 0 & -0.5995 \\ -0.8074 & -0.3484 & -0.3302 & 0.2116 & -0.6806 & 0.9966 & 0.5995 & 0 \end{bmatrix} \tag{28}$$

and a symmetric damping matrix given by

$$D = \begin{bmatrix} 0.2166 & -0.1567 & -0.0184 & 0.0217 & 0.0311 & -0.0897 & -0.0159 & 0.0830 \\ -0.1567 & 0.1915 & 0.0675 & -0.0385 & 0.0270 & 0.0171 & -0.0081 & -0.1581 \\ -0.0184 & 0.0675 & 0.3568 & -0.0562 & 0.0919 & -0.0079 & 0.0020 & -0.1075 \\ 0.0217 & -0.0385 & -0.0562 & 0.1068 & 0.0314 & 0.0698 & -0.0119 & -0.0524 \\ 0.0311 & 0.0270 & 0.0919 & 0.0314 & 0.1170 & -0.0273 & -0.0844 & -0.1121 \\ -0.0897 & 0.0171 & -0.0079 & 0.0698 & -0.0273 & 0.1733 & 0.0082 & -0.0237 \\ -0.0159 & -0.0081 & 0.0020 & -0.0119 & -0.0844 & 0.0082 & 0.1897 & -0.0144 \\ 0.0830 & -0.1581 & -0.1075 & -0.0524 & -0.1121 & -0.0237 & -0.0144 & 0.3513 \end{bmatrix} \tag{29}$$

For brevity, we show these matrices with numbers only up to 4 decimal figures. A small computational exercise confirms that the two necessary and sufficient conditions given in Eqs. (15) and (16) are satisfied by K and G. Hence, Result 2 is applicable.

The orthogonal matrix

$$Q = \begin{bmatrix} -0.2791 & -0.4238 & -0.3262 & 0.5234 & 0.3017 & 0.0774 & 0.0137 & 0.5148 \\ 0.1341 & -0.7320 & 0.4453 & -0.2478 & -0.2947 & 0.2742 & 0.0817 & 0.1335 \\ 0.4570 & -0.0001 & 0.4705 & 0.5918 & 0.0850 & -0.4020 & 0.2194 & -0.0512 \\ -0.4716 & -0.3072 & -0.0061 & -0.1164 & 0.2458 & -0.3252 & 0.5009 & -0.5026 \\ -0.3623 & -0.1491 & 0.2283 & 0.3237 & -0.1337 & -0.0349 & -0.7139 & -0.4010 \\ -0.0602 & -0.0938 & 0.0924 & -0.3914 & 0.0869 & -0.7409 & -0.3111 & 0.4154 \\ 0.3948 & -0.1893 & -0.0153 & -0.1980 & 0.7784 & 0.1783 & -0.2833 & -0.2257 \\ 0.4268 & -0.3512 & -0.6427 & 0.0605 & -0.3509 & -0.2636 & -0.0861 & -0.2788 \end{bmatrix}$$

whose columns are the orthonormal eigenvectors of the matrix K then yields the matrices

$$\Lambda = Q^T KQ = \text{diag}(3.5, 0.5, 4.0, 2.5, 3.0, 1.0, 1.5, 2.0)$$

$$\Gamma = Q^T GQ = \text{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0, 0\right)$$

and

$$D_{\text{diag}} = Q^T DQ = \text{diag}(0.2754, 0.0144, 0.6240, 0.3671, 0.2050, 0.1910, 0.0041, 0.022)$$

so that the system is quasi-diagonalized (and uncoupled) to the form shown in Eq. (17).

We see, therefore, that Eq. (2) with the matrices K, G, and D given in Eqs. (27)–(29) is decomposed into three independent two-degree-of-freedom damped gyroscopic potential subsystems and two independent single-degree-of-freedom damped potential subsystems; the response of each of these subsystems can, of course, be separately obtained.

As required by Result 2, the symmetric matrix D *must* be expressible in the form shown in Eq. (7). It is easy to verify that, as required

$$D = 0.001(-K - 2K^2 + 2K^4 + (GKG)^2 + 3G^4) \quad \blacksquare$$

Remark 3. By using Eqs. (3)–(6), all the Results, Corollaries, and Remarks, for the system described by Eq. (2) can be translated for Eq. (1).

For example, in the original dynamical system described by Eq. (1) with matrices $\tilde{M}$, $\tilde{K}$, $\tilde{G}$, and $\tilde{D}$, Eq. (7) becomes

$$\tilde{D} = u_0\tilde{M} + \tilde{M}\sum_{i=1}^{n-1}(a_i(\tilde{M}^{-1}\tilde{K})^i + b_i(\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1}\tilde{G})^i) + \tilde{M}\sum_{i=1}^{h-1}c_i(\tilde{M}^{-1}\tilde{G})^{2i} \tag{30}$$

and the uncoupling conditions under which the system reduces to the form (17)–(20) by a real change of coordinates are equivalent to the conditions

$$\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1}\tilde{K} \tag{31}$$

and

$$(\tilde{K}\tilde{M}^{-1}\tilde{G}\tilde{M}^{-1})^2 = (\tilde{G}\tilde{M}^{-1}\tilde{K}\tilde{M}^{-1})^2 \tag{32}$$

Also, the condition $KG = GK$ in Corollary 2 becomes

$$\tilde{K}\tilde{M}^{-1}\tilde{G} = \tilde{G}\tilde{M}^{-1}\tilde{K} \tag{33}$$

$\blacksquare$

3 Conclusions

This paper investigates the uncoupling of an MDOF damped gyroscopic potential system through the use of a real coordinate transformation into independent subsystems, each of which has at most two degrees-of-freedom. This brings about a reduction of the response of such a damped MDOF system to the response of subsystems of much lower order. It, therefore, provides an improved understanding of the physics of the complex behavior of such an MDOF system, affords closed-form solutions to the response of such a system to external forces, and yields efficient and more accurate computational methods in the determination of its response.

From an application viewpoint, the following conclusions can be drawn from this study.

(a) Two independent necessary and sufficient conditions are shown to be required for such an uncoupling to occur when using the special form of the damping matrix considered here.
(b) When the nonzero eigenvalues of the gyroscopic matrix are distinct, a situation that arises often in practice, these two independent conditions reduce to a single necessary and sufficient condition for such uncoupling to be guaranteed.
(c) When the potential (stiffness) matrix and the gyroscopic matrix commute the uncoupling is also shown to be guaranteed. However, from a practical standpoint, this result appears not to be very useful since it demands that the stiffness matrix must have many eigenvalues with multiplicity greater than unity—a situation that is generally uncommon in naturally occurring and engineered systems. The two independent conditions obtained in item (a) above are satisfied by a much broader range of stiffness and gyroscopic matrices and, therefore, are more useful for real-life applications.
(d) It is shown that when the potential (stiffness) matrix has distinct eigenvalues—a situation that commonly arises in civil, mechanical, and aerospace engineering, and also in nature—*every* damping matrix that permits the uncoupling of a damped gyroscopic potential MDOF system into independent subsystems with at most two degrees-of-freedom *must* have the special form of the damping matrix presented herein.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

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